

Remarks on $2 + 1$ Self-dual Chern-Simons Gravity

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Abstract

We study $2 + 1$ Chern-Simons gravity at the classical action level. In particular we rederive the linear combinations of the “standard” and “exotic” Einstein actions, from the (anti) self-duality of the “internal” Lorentzian indices. The relation to a genuine four-dimensional (anti)self-dual topological theory greatly facilitates the analysis and its relation to hyperbolic three-dimensional geometry. Finally a non-abelian vector field “dual” action is also obtained.

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I. INTRODUCTION

$2 + 1$ general relativity has many remarkable features such as renormalizability, integrability, topological invariance etc., which have been of great value as an heuristic guide in the quest of the correct theory of the quantum gravitational field in four dimensions (for a review, see [1]).

These properties derive from the fact that $2+1$ Einstein gravity can be written as a Chern-Simons gauge theory. This formulation is deeply related to the celebrated BTZ black hole solution [2] and recently this relation has been of great importance in the further development of the AdS/CFT correspondence [3], in particular in the $\text{AdS}_3/\text{CFT}_2$ correspondence in the context of string theory [4]. In the AdS_3 case, in the context of topological field theories, this correspondence can be also realized as the well-known correspondence between a $2 + 1$ Chern-Simons gravity theory on the bulk three-manifold X and the rational conformal field theory WZW_2 on the boundary of this bulk (that is the CS_3/WZW_2 correspondence) [5].

In Refs. [6,7] it was found that general relativity in $2 + 1$ dimensions with cosmological constant λ is equivalent to a pure Chern-Simons gauge theory on an arbitrary three-manifold X and certain gauge group \mathbf{G} . X is locally homogeneous with curvature equal to λ and for definiteness we will take it to be given by $\Sigma \times \mathbf{R}$ with Σ a Riemann surface. For $\lambda > 0$, the gauge group \mathbf{G} is the de Sitter group $\text{SO}(3, 1)$; for $\lambda < 0$, it is the anti-de Sitter group $\text{SO}(2, 2)$ and for $\lambda = 0$, it is the Poincaré group $\text{ISO}(2, 1)$. In Ref. [7] Witten showed, that there are two actions associated to two different ways of parametrizing the invariant quadratic form on the Lie algebra of \mathbf{G} ; namely, the “standard” Einstein action which characterize classically gravity in $2 + 1$ dimensions and the “exotic” one. It is interesting to note that each of these actions gives the same classical equations, but their quantization is expected to lead to two different quantum theories. Topology change of space-time (with $\lambda = 0$) was subsequently discussed within this context [8]. Some further developments of Chern-Simons supergravity and extended supergravity were considered in [9]. Applications of Chern-Simons gravity and supergravity to the two particle scattering problem were studied in [10].

In this paper we will follow the philosophy of MacDowell and Mansouri (MM) [11] and apply it to the 2+1 Chern-Simons gravity. Within this philosophy, the Chern-Simons action depends only on the gauge field of the group under consideration. With the usual diagonal Cartan group metric and the identification of its components with the spin connection and the vierbein we obtain the “exotic” action .

This 2 + 1-MM type of action is then naturally generalized by taking its (anti)self-dual part with respect to the internal indices of the gauge field. The procedure followed is similar to that in a previous work [12], in which the 3 + 1-MM action was generalized for the (anti)self-dual gauge field. It was shown there that this model is a generalization of the Plebański-Ashtekar dynamical action [13]. We show that the two different classical actions, the “standard” and the “exotic” ones, describing Chern-Simons gravity, are encoded in the (anti)self-duality of the internal indices of the structure group \mathbf{G} of the frame bundle. This (anti)self-dual theory corresponds, in 3+1 dimensions, to both the MM self-dual action [12] and the Plebański-Ashtekar formulation [13].

Linear combinations of the “exotic” and the “standard” actions have been already considered in the literature [6,7,14–16]. However, for this pure topological 2 + 1 Chern-Simons theory, these two actions appear together by considering the (anti)self-dual action. It has been previously shown, that for a pure 3+1 topological gravitational model [17,18], the same kind of procedure gives the sum of the Euler characteristic $\chi(X)$ plus $i\sigma(X)$, with $\sigma(X)$ being the signature, in a way the Euler characteristic seems to play the role of a “ θ -term” [19] (Being both terms topological one could also say that $\sigma(X)$ is a “ θ -term”). Similarly, here the “standard” (or “exotic”) action will play the role of a “ θ -term”. It is well known that the precise relation between the classical topological invariants in four dimensional manifolds W and the Chern-Simons invariant CS in its non-trivial boundary $\partial W = X$ is through the Atiyah-Patodi-Singer formula [20]

$$\sigma(W) = \int_W P_1(W) - \int_X CS - \frac{1}{2}\eta_S, \quad (1)$$

where $P_1(W)$ is the first Pontrjagin class of W *i.e.* $\int_W Tr(R \wedge R)$, CS is the Chern-Simons

action and η_S is the η -invariant of X . The restriction to self-dual (or anti-self-dual) part of the curvature in W , with respect to the group indices, leads to the Atiyah-Patodi-Singer formula of the form

$$\sigma(W^\pm) = \int_{W^\pm} P_1(W^\pm) - \int_X CS^\pm - \frac{1}{2}\eta_S^\pm, \quad (2)$$

where W^\pm denotes the space W with self-dual (or anti-self-dual) curvature tensor. We will find in Sec. III that keeping this notion of (anti)self-duality some set of known data find a better understanding with these data correlated by this (anti)self-duality.

It is known from [7] that some quantum aspects of Chern-Simons gravity are related to the mathematical description of hyperbolic geometry of the three-manifold X [21–23]. In fact it was shown in [7] that the volume and Chern-Simons invariants V and CS , are related to the “standard” and “exotic” actions respectively. However in order to distinguish hyperbolic three-manifolds X with the same invariants V and CS , a more refined invariant is needed, the η -invariant [24,25]. In this work we exhibit that the η -invariant is originated, in the Chern-Simons gravity theory, from its relation to an (anti)self-dual topological gravity theory in four dimensions. Thus all invariants V , CS and η_S are encoded in a self-dual topological gravity theory in four dimensions.

In previous works [17,18], we were able to exhibit “field theory duality” actions, where the dual action appears with inverted coupling constants (for a review of duality in field and string theory, see for example [26]). This was performed for a pure topological theory and for 3 + 1 MM-gravity and supergravity. For Chern-Simons gravity, a “field theory duality” will be found in this paper, in the sense of Rocek and Verlinde [27], which also gives for the dual action a non linear sigma-model of the Freedman-Townsend type [28], that in this case can be further integrated to a Chern-Simons action with inverted coupling constant.

This work is organized as follows: In section II we give an overview of 2+1 Chern-Simons gravity. In section III the consideration of the (anti)self-dual spin-connection (with respect to the internal “Lorentzian” indices) leads to the “standard” and “exotic” terms, which emerge as a linear combination. Also in this same section the correspondence with some

aspects of hyperbolic geometry is described. A “dual” theory to Chern-Simons gravity is presented in section IV. In section V we finally give our concluding remarks.

II. OVERVIEW OF 2 + 1 CHERN-SIMONS GRAVITY

In this section we will briefly recall the basic structure of 2 + 1 Chern-Simons gravity and the relevant structure of the fields, Lagrangians and symmetries which we will need in the following sections. For a more complete treatment see Refs. [1,7]. In particular we will follow the notation of these references.

In [6,7] it was shown that the 2 + 1 Einstein-Hilbert action with vanishing cosmological constant is equivalent to a Chern-Simons action in 2 + 1 dimensions with gauge group given by the Poincaré group ISO(2,1).

In the case of nonvanishing cosmological constant $\lambda \neq 0$, Witten found that the natural generalization is given by the “standard” Einstein action [7]

$$I = \int_X \varepsilon^{ijk} \left(e_{ia} (\partial_j \omega_k^a - \partial_k \omega_j^a) + \varepsilon_{abc} e_i^a \omega_j^b \omega_k^c + \frac{1}{3} \lambda \varepsilon_{abc} e_i^a e_j^b e_k^c \right). \quad (3)$$

This Einstein-Hilbert action gives rise to two spacetimes, one for $\lambda > 0$ whose covering is a portion of the de Sitter space with symmetry group SO(3,1) and the other one with $\lambda < 0$ with symmetry group being SO(2,2). There is another general action, which can be constructed by taking the standard diagonal representation of the invariant quadratic form of the Lie algebra of the gauge group [7]. This action was termed the “exotic” action and it is given by

$$\tilde{I} = \int_X \varepsilon^{ijk} \left(\omega_i^a (\partial_j \omega_k^a - \partial_k \omega_j^a) + \frac{2}{3} \varepsilon_{abc} \omega_j^b \omega_k^c + \lambda e_i^a (\partial_j e_k^a - \partial_k e_j^a) + 2 \lambda \varepsilon_{abc} \omega_i^a e_j^b e_k^c \right). \quad (4)$$

This action has the same classical symmetries as the “standard” one and for this reason it can be added to the “standard” Einstein action. Classically, both “standard” and “exotic” actions are equivalent. In a representation of the Lie algebra where its generators are given

by (J_a^+, J_a^-) where $J_a^\pm = \frac{1}{2}(J_a \pm \frac{1}{\sqrt{\lambda}}P_a)$ for $\lambda \neq 0$, the connections are expressed as ${}^\pm A_i = {}^+A^a J_a^+ \pm {}^-A^a J_a^-$ where ${}^\pm A_i^a = \omega_i^a \pm \sqrt{\lambda}e_i^a$. For $\lambda < 0$, the group $SO(2, 2)$ is isomorphic to $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ and consequently it undergoes a splitting over the real numbers. For $\lambda > 0$, the Lie group $SO(3, 1)$ does split as well when complexified, to $SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$.

The Chern-Simons action reads then

$$I^\pm = \int_X \varepsilon^{ijk} (2^\pm A_i^a \partial_j {}^\pm A_k^a + \frac{2}{3} \varepsilon_{abc} {}^\pm A_i^{a\pm} A_j^{b\pm} A_k^c). \quad (5)$$

In terms of the I^\pm the “standard” and “exotic” actions are written respectively as

$$I = \frac{1}{2\sqrt{\lambda}}(I^+ - I^-), \quad \tilde{I} = \frac{1}{2}(I^+ + I^-). \quad (6)$$

Under an euclidean continuation to go from Minkowski space-time to the euclidean one, the “standard” term is real and the only possible modification (for renormalizing purposes) is a rescaling of the vierbein by the constant \hbar . Moreover, the “exotic” term should arise as pure imaginary and the parameter k in its coefficient should be quantized. Finally it is worth to mention that one can define the euclidean partition function

$$Z(X) = \int \mathcal{D}e \mathcal{D}\omega e^{-\hat{I}}, \quad (7)$$

of the appropriate linear combination of the “exotic” and “standard” actions to be

$$\hat{I} = \frac{1}{\hbar}I + i\frac{k}{8\pi}\tilde{I}, \quad (8)$$

with $k \in \mathbf{Z}$. In the limit $\hbar \rightarrow 0$, $Z(X)$ could “flow” towards a critical point where the partition function would be described by two geometric invariants of the hyperbolic geometry of three-manifolds: the volume V and the Chern-Simons invariant CS . These invariants are described by the “standard” and the “exotic” actions respectively.

III. THE CASE OF THE SELF-DUAL SPIN CONNECTION

In this section we will work out the Chern-Simons Lagrangian for (anti)self-dual gauge connection with respect to duality transformations of the internal indices of the gauge group

\mathbf{G} , in the same philosophy of MM [11], and of [12]. We will show that the two actions “standard” and “exotic” arise in a natural manner. The action is given by

$$L^\pm_{CS} = \int_X \varepsilon^{ijk} \left(\pm A_i^{AB} \partial_j^\pm A_{kAB} + \frac{2}{3} \pm A_{iA}^B \pm A_{jB}^C \pm A_{kC}^A \right), \quad (9)$$

where $A, B, C, D = 0, 1, 2, 3$, $\eta_{AB} = \text{diag}(-1, +1, +1, +1)$ and the complex (anti) self-dual connections are

$$\pm A_i^{AB} = \frac{1}{2} (A_i^{AB} \mp \frac{i}{2} \varepsilon_{CD}^{AB} A_i^{CD}). \quad (10)$$

Which satisfy the relation $\varepsilon_{CD}^{AB} \pm A_i^{CD} = \pm i \pm A_i^{AB}$.

We can compute the first term of the right hand side of the action (9) and this gives

$$\varepsilon^{ijk\pm} A_i^{AB} \partial_j^\pm A_{kAB} = \varepsilon^{ijk} \left(\frac{1}{2} A_i^{AB} \partial_j A_{kAB} \mp \frac{i}{2} \varepsilon^{ABCD} A_{iAB} \partial_j A_{kCD} \right), \quad (11)$$

while the second part gives

$$\varepsilon^{ijk\pm} A_{iA}^{B\pm} A_{jB}^{C\pm} A_{kC}^A = \frac{1}{2} \varepsilon^{ijk} (A_{iA}^B A_{jB}^C A_{kC}^A \pm \frac{i}{2} \varepsilon^{ABCD} A_{iA}^E A_{jEB} A_{kCD}). \quad (12)$$

Thus using Eqs. (11) and (12), the action (9) can be rewritten as

$$L^\pm_{CS} = \int_X \frac{1}{2} \varepsilon^{ijk} \left(A_i^{AB} \partial_j A_{kAB} + \frac{2}{3} A_{iA}^B A_{jB}^C A_{kC}^A \right) \mp \frac{i}{4} \varepsilon^{ijk} \varepsilon^{ABCD} \left(A_{iAB} \partial_j A_{kCD} + \frac{2}{3} A_{iA}^E A_{jEB} A_{kCD} \right). \quad (13)$$

In this expression the first term is the Chern-Simons action for the gauge group \mathbf{G} , while the second term appears as its corresponding “ θ -term”. The same result was obtained in 3+1 dimensions when we considered the (anti)self-dual MM action [18], or the (anti)self-dual 3 + 1 pure topological gravitational action [17]. It is interesting to note that the “ θ -term” in these theories arise by means of the gauge group indices, taking the role that usually the space-time indices have in the abelian [19] and non-abelian cases. Normally for gauge theories it is meaningful to have arbitrary coupling constants for the dynamical and the θ -terms (as in the 3 + 1-d case [18]), for that purpose we need a linear combination of the self-dual and antiself-dual actions (9). This matter is further discussed in the next section.

One should remark that the two terms in the action (13) are the Chern-Simons and the corresponding “ θ -term” for the gauge group \mathbf{G} under consideration. After imposing the particular identification $A_i^{AB} = (A_i^{ab}, A_i^{3a}) = (\omega_i^{ab}, \sqrt{\lambda}e_i^a)$ and $\omega_i^{ab} = \varepsilon^{abc}\omega_{ic}$, the “exotic” and “standard” actions for the gauge group $SO(3, 1)$ are given by

$$L_{CS}^{\pm} = \int_X \frac{1}{2} \varepsilon^{ijk} \left(\omega_i^a (\partial_j \omega_{ka} - \partial_k \omega_{ja}) + \frac{2}{3} \varepsilon_{abc} \omega_i^a \omega_j^b \omega_k^c + \lambda e_i^a (\partial_j e_{ka} - \partial_k e_{ja}) - 2\lambda \varepsilon_{abc} e_i^a e_j^b e_k^c \right) \\ \pm i \sqrt{\lambda} \varepsilon^{ijk} \left(e_i^a (\partial_j \omega_{ka} - \partial_k \omega_{ja}) - \varepsilon_{abc} e_i^a \omega_j^b \omega_k^c + \frac{1}{3} \lambda \varepsilon_{abc} e_i^a e_j^b e_k^c \right), \quad (14)$$

plus surface terms. These actions degenerate at $\lambda = 0$, giving the pure $SO(2,1)$ Chern-Simons action. However, we can take a linear combination of them $\tau L_{CS}^+ + \bar{\tau} L_{CS}^-$. In this way, the $\sqrt{\lambda}$ in front of the “standard” term in (14) can be absorbed by a redefinition of the imaginary part of τ . Thus, with this linear combination, the $\lambda = 0$ case gives us the Einstein 2 + 1 gravity without cosmological constant, plus a topological $SO(2, 1)$ CS term.

This action evidently can be rewritten in terms of the “standard” I and the “exotic” \tilde{I} actions reviewed at section II, Eq.(8) as

$$L_{CS}^{\pm} = \frac{1}{2} \tilde{I} \pm iI. \quad (15)$$

For $SO(2, 2)$ the same procedure can be followed. In this case the (anti) self-dual connections A_i^{\pm} are real, hence the action (14) is also real, as can be obtained by setting $\lambda \rightarrow -\lambda$. Thus, in (14), both cases $\lambda > 0$ and $\lambda < 0$ are encoded in the same framework, which permits a simple analysis of three-dimensional hyperbolic geometry.

Up to here we have worked with the Minkowski signature. We can perform a Wick rotation and the (anti)self-dual Chern-Simons action (15) will be modified by a global factor i , so the above action can be written as

$$L_{CS}^{\pm} = I \pm i \frac{1}{2} \tilde{I}. \quad (16)$$

This is precisely the relevant linear combination of the “standard” and “exotic” actions used to describe the hyperbolic geometry of X in the euclidean case, through the volume form V

and the Chern-Simons invariant CS [7,21–23]. Thus the euclidean partition function of the (anti)self-dual Chern-Simons action (9) is itself a topological invariant

$$Z(X) = \exp\left(\frac{V}{\hbar} \pm 2\pi i k CS\right). \quad (17)$$

This provides a more refined invariant than simply using the volume invariant V and it has been proved to be a complex analytic function in the space of smooth deformations in the space of the hyperbolic structures on X [22,23]. However, there is another natural invariant, as we have seen, which is the η -invariant which does not enter in the formula (17). The η -invariant is very important since it permits to distinguish two hyperbolic three-manifolds with the same volume V and CS invariants, that is, with the same partition function $Z(X)$ [24].

At this point one would be worried about the (anti)self-duality for the Chern-Simons gauge theory because this concept is well defined only in four dimensions. This can be easily solved by noting from Eq. (2) that the (anti)self-dual Chern-Simons action (9) can be naturally related to the classical topological invariants (Euler and signature) of the four dimensional spaces W^\pm and the η -invariant. Thus the (anti)self-duality used in this section is induced by a genuine (anti)self-duality from the theory in four dimensions. Moreover the needed η -invariant can be extracted from the stationary phase evaluation of (9) following [5].

There is another issue, which can be better understood also with the idea of the (anti)self-duality in the internal indices. Euclidean and Lorentzian quantum field theory of the Chern-Simons gravity Lagrangians, L and \bar{L} differ by a factor i as we have seen above. From the point of view of the geometry of X the change of the Lagrangians are encoded in a change of the orientation of X . From the pure topological Chern-Simons gravity point of view both Lagrangians are equivalent because the theory is independent of the metric of X . This is not satisfactory because both actions amount to very different quantum theories. It finds a nice explanation by using the (anti)self-duality inherited by the four-dimensional theory on W . The reason is as follows: a change of the orientation of the bulk W i.e.

$e_0 \wedge e_1 \wedge e_2 \wedge e_3 \rightarrow -e_0 \wedge e_1 \wedge e_2 \wedge e_3$ leads to a change of orientation on the boundary $e_0 \wedge e_1 \wedge e_2 \rightarrow -e_0 \wedge e_1 \wedge e_2$, for fixed e_3 . In four dimensions the (anti)self-dual projection implies the choice of an orientation on W . Thus as we have seen this choice is preserved on the boundary X of W and the choice of L or \bar{L} corresponds to the choice of L_{CS}^\pm in Eq. (9).

Some other new invariants are indeed necessary in order to describe different hyperbolic three-manifolds with the three classical invariants equal [25]. The description of these invariants from the Chern-Simons gravity and supergravity point of view, deserves further research.

IV. CHERN-SIMONS GRAVITY DUAL ACTION

This section is devoted to show that a “dual” action to the Chern-Simons gravity action can be constructed following [27]. It is worth to mention that for the abelian case a dual action to the Chern-Simons action that does invert the coupling constant was worked out previously by Balachandran *et al*, in the context of the quantum Hall effect theory [29].

Before we proceed to show the “duality” of the Chern-Simons gravity action under vector field transformations we would like to describe our motivation for it. If one substitutes Eq. (6) in the linear combination (8) we get

$$\hat{I} = \tau^+ I^+ - \tau^- I^- \quad (18)$$

with

$$\tau^\pm = \frac{1}{2\sqrt{\lambda\hbar}} \pm i \frac{k}{16\pi}. \quad (19)$$

Equation (18) has the typical form of actions transforming under modular transformations $SL(2, \mathbf{Z})$ [19]. Clearly the first term in Eq. (19) is the coupling constant of the gravity theory (6) while the second term is a kind of θ -term. Thus one can ask whether there exists some modular transformation which transforms the partition function (7)

$$Z\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^u (c\bar{\tau} + d)^v Z(\tau), \quad (20)$$

where u and v are the weights of the modular transformation. This would have importance in the construction of further new invariants in the description of hyperbolic geometry as we have seen at the end of section III. In this section we will show that such a transformation exists but action (18) is in fact self-dual under this $\text{SL}(2, \mathbf{Z})$ group.

We begin from the original non-abelian Chern-Simons action given by

$$L = \int_X \frac{g}{4\pi} \text{Tr}(A \wedge H), \quad (21)$$

where $H = dA + \frac{2}{3}A \wedge A$ and Tr is a quadratic form in the Lie algebra of the relevant gauge group \mathbf{G} . In what follows we will use the usual conventions: $A = A_i^{AB} M_{AB} dx^i$, $G = G_{ij}^{AB} M_{AB} dx^i \wedge dx^j$, where M_{AB} are the generators of the Lie algebra of \mathbf{G} and they satisfy $[M_{AB}, M_{CD}] = i f_{ABCD}^{EF} M_{EF}$ with f_{ABCD}^{EF} the structure constants of the Lie algebra. The invariant quadratic form on the Lie algebra is given by $\text{Tr}(M_{AB} M_{CD}) = \eta_{(AC} \eta_{BD)}$. With these conventions, the above action is written as

$$L = \int_X d^3x \frac{g}{4\pi} \varepsilon^{ijk} A_i^{AB} \left(\partial_j A_{kAB} + \frac{1}{3} f_{AB CDE F} A_j^{CD} A_k^{EF} \right). \quad (22)$$

Now, as usual we propose a parent action in order to derive the dual action to (22),

$$L_D = \int_X a \text{Tr}(B \wedge H) + b \text{Tr}(A \wedge G) + c \text{Tr}(B \wedge G), \quad (23)$$

where as usual, B and G are Lie algebra-valued one and two forms over X and a, b, c are constants to be determined. In local coordinates on X , the above action can be written as

$$L_D = \int_X d^3x \varepsilon^{ijk} \left(a B_i^{AB} H_{jkAB} + b A_i^{AB} G_{jkAB} + c B_i^{AB} G_{jkAB} \right), \quad (24)$$

where

$$H_{jkAB} = \partial_j A_{kAB} + \frac{1}{3} f_{AB CDE F} A_j^{CD} A_k^{EF}. \quad (25)$$

It is a very easy matter to show that Eq. (22) can be derived from the parent action. To see that, consider first the partition function of the parent action of the form

$$Z = \int \mathcal{D}A \mathcal{D}G \mathcal{D}B \exp(+iL_D), \quad (26)$$

$$\exp(+i\tilde{L}_D) = \int \mathcal{D}G \mathcal{D}B \exp\left(+i \int_X d^3x \varepsilon^{ijk} (a B_i^{AB} H_{jkAB} + b A_i^{AB} G_{jkAB} + c B_i^{AB} G_{jkAB})\right). \quad (27)$$

One can integrate out first with respect G . This gives

$$\exp(+i\tilde{L}_D) = \int \mathcal{D}B \delta(b A_i^{AB} + c B_i^{AB}) \exp\left(+ia \int_X d^3x \varepsilon^{ijk} B_i^{AB} H_{jkAB}\right). \quad (28)$$

Integration with respect to B gives the final form

$$\tilde{L}_D = -\frac{ab}{c} \int_X d^3x \varepsilon^{ijk} A_i^{AB} (\partial_j A_{kAB} + \frac{1}{3} f_{AB CDEF} A_j^{CD} A_k^{EF}). \quad (29)$$

A choice of the constants of the form

$$c = -\frac{4\pi}{g}, \quad a = b = 1, \quad (30)$$

immediately gives the formula (22).

The “dual” action L_D^* can be computed as usually in the euclidean partition function, by integrating first out with respect to the physical degrees of freedom A

$$\exp(-L_D^*) = \int \mathcal{D}A \exp(-L_D). \quad (31)$$

The resulting action is of the gaussian type in the variable A and thus after some computations it is easy to find the “dual” action

$$L_D^* = \int_X d^3x \varepsilon^{ijk} \left\{ -\frac{3}{4a} (a \partial_i B_{jAB} + b G_{ijAB}) [\mathbf{R}^{-1}]_{kn}^{ABCD} \varepsilon^{lmn} (a \partial_l B_{mCD} + b G_{lmCD}) + c \alpha_i^{AB} G_{jkAB} \right\}, \quad (32)$$

where $[\mathbf{R}]$ is given by $[\mathbf{R}]_{ABCD}^{ij} = \varepsilon^{ijk} f_{ABCD}^{EF} B_{kEF}$ whose inverse is defined by $[\mathbf{R}]_{ABCD}^{ij} [\mathbf{R}^{-1}]_{jk}^{CDEF} = \delta_k^i \delta_{AB}^{EF}$.

The partition function is finally given by

$$Z = \int \mathcal{D}G \mathcal{D}B \sqrt{\det(\mathbf{M}^{-1})} \exp(-L_D^*). \quad (33)$$

In this “dual action” the G field is not dynamical and can be integrated out. If we take the values (30) for the constants of the parent action (23), then the integration of this auxiliary field gives

$$Z = \int \mathcal{D}B \exp\left(-\frac{4\pi}{g}\varepsilon^{lmn}(B_l^{AB}\partial_m B_{nAB} - \frac{4\pi}{g}f_{ABCDEF}B_l^{AB}B_m^{CD}B_n^{EF})\right). \quad (34)$$

As Balachandran et. al. mention, in the abelian case [29], the fields B cannot be rescaled if we impose “periodicity” conditions on them. Thus, this dual action has inverted coupling with respect to the original one.

V. CONCLUDING REMARKS

In this paper we have rederived linear combinations of the “standard” and “exotic” actions given by Witten [7]. An interesting feature is that both actions are encoded in the (anti)self-duality of the gauge group \mathbf{G} of the theory. Thus both actions are obtained from the Chern-Simons gravity action for the (anti)self-dual connection (9) of \mathbf{G} . The usual Chern-Simons action corresponds to the “exotic” one and its “ θ -term” gives the “standard” action. So to construct a linear combination of the Chern-Simons action and its “ θ -term” is equivalent to a linear combination of the self-dual and antiself-dual actions, as we would have expected.

As a matter of fact, we have provided the justification of this derivation by observing that the linear combinations of the actions I and \tilde{I} come from the (anti)self-duality structure of the four manifold W when one is restricted to work on the boundary X of W .

Furthermore, for the (non-abelian) Chern-Simons gravity action, we have found a “non-abelian dual action” in the sense of [27]. This resulting “dual action” consists of a non linear sigma model [28] of the Freedman-Townsend type eq. (32). Moreover, after a further integration, one can reduce this action to a Chern-Simons action with inverted coupling eq. (34). It is interesting to remark that all these calculations are performed at the level of the

gauge fields A_i^{AB} and its corresponding Chern-Simons action with a “ θ -term”. It is only when we identify with gravity, that we come to the “standard” and “exotic” actions and to a metrical theory.

The results obtained in section III can be easily generalized to Chern-Simons supergravity. It is tantalizing to find the supersymmetric extension of the “dual” action obtained in section IV [18,30]. Furthermore it would be of some interest to apply the results of this paper to explore further uses of Chern-Simons gravity to hyperbolic geometry of X and perturbative expansions of Chern-Simons theory with non-compact groups [31]. Work in these directions will be reported elsewhere.

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